On Double Spend Races

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Mathematical Fondation of Bitcoin

Article Double Spend Races, in collaboration with Ricardo Perez-Marco
arXiv:1702.02867 [cs.CR]

Satoshi Risk Tables, arXiv:1702.04421 [cs.CR]

Section 11. Calculations of Bitcoin: A Peer-to-Peer Electronic Cash System, Satoshi Nakamoto, 2008.

Following a previous work by Meni Rosenfeld
Analysis of hashrate-based double-spending, 2012

- Correction of Satoshi's calculus for the probability of success of a double spend attack
- Proof that "the probability drops exponentially as the number of blocks the attacker has to catch up with increases" (Satoshi)
- Closed form formula with Beta function for this probability
- More accurate risk analysis knowing the time it took to validate blocks.
- Underestimation of the probability of double-spend attack

Two groundbreaking ideas in Bitcoin

- New framework for the design of a transaction
- Breakthrough in distributed system theory

Concept of "smart-contract" (prophetized by Nick Szabo)
ScriptSig / ScriptPubKey (not in the white paper)

Use of proof-of-work (rediscovered by Adam Back) to create a decentralized blockchain

No bibliography at all related with the distributed system theory!

Main references in cryptography (Haber& Stornetta for timestamps server)

Variation of two generals problem. Fisher, Lynch et Paterson, 1985

Theorem. *In a asynchronous model, there is no deterministic algorithm to achieve consensus (if at least one node can crash)*

However, there are randomized consensus.

Randomization makes algorithm powerful...

Proof-of-Work

Time consuming

Cost function. A string, D integer, x integer

$$\begin{array}{rrcl} \mathcal{F}\colon & \mathcal{C} \times [0, D_{\max}] \times [0, N] & \longrightarrow & \{\text{True}, \text{False}\} \\ & (A, D, x) & \longmapsto & \mathcal{F}(A, D, x) \end{array}$$

Problem. Given A (string) and D (level of difficulty), find $\mathbf{x}$ such that

$$\mathcal{F}(A, D, \mathbf{x}) = \text{True} \quad (1)$$

Solution $\mathbf{x}$ (not necessarily unique) is a "proof-of-work" called **nonce**. Problem possibly hard to solve. Use of computational power to solve it.

Pricing via Processing or Combatting Junk Mail, C. Dwork and M. Naor, (1993).
Denial-of-service counter measure technique in a number of systems
Anti-spam tool

Hashcash, A Denial of Service Counter-Measure, A. Back, preprint (2002)
Hashcash: a proof-of-work algorithm
Create a stamp to attach to mail
Cost functions proposed are different
Solution of (1) by brute-force.

Hash functions

Use of hash function h to create a puzzle
Example: $\mathcal{F}(A, D, x) = \text{True}$ if $h(A|x)$ starts with D zeros and false else.

Rabin, Yuval, Merkle, late 70.
"Swiss army knife" of cryptography

- input of any size
- output of fixed-size
- easy to calculate (in $O(n)$ if input is n-bit string)

i. collision resistance

ii. preimage resistance

iii. second preimage resistance

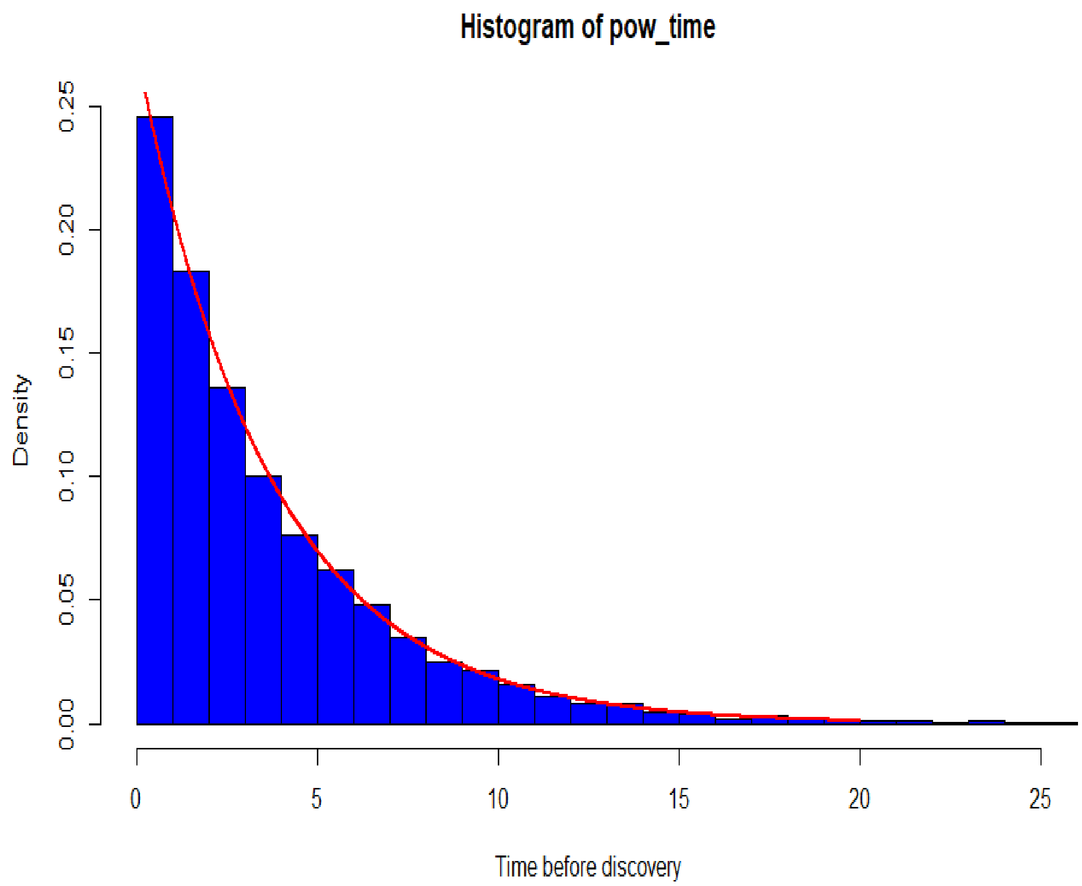
One way function
Random Oracles are Practical: A Paradigm for Designing Efficient Protocols, M. Bellare, P. Rogaway, ACM Conference on Computer and Communications Security (1993).
Based on block ciphers
Compression function
Merkle–Damgård construction
Message digest
Commitments
Puzzle
Digital signature
SHA-1, MD5 broken
SHA-2

Test of SHA256

Images are uniform & Easy to compute

Proposition. *If h is a hash function, then the time of resolution before getting a "proof-of-work" for a problem of difficulty D has an exponential distribution.*

Example. Problem: find x such that SHA256($a|x$) starts with 4 zeros with a an arbitrary string. Sample (τ_i). Mean $\approx$ 4 sec.



However, it is not clear that the distribution is exponential. Tests Cramer-von-Mises and Kolmogorov-Smirnov fail if size(sample)>6000 with R software...

Interblock times

Hash function $h = \text{SHA256} \circ \text{SHA256}$

$$
\begin{aligned}
\mathcal{F}(A, D, \mathbf{x}) &= \mathbb{1}_{h(A|D|x) < \frac{2^{224}}{D}} \\
A &= x_1 | x_2 | x_3 | x_4 | \\
x_1 &= \text{Version} \\
x_2 &= \text{Hash Previous Block} \\
x_3 &= \text{Hash Merkle Root} \\
x_4 &= \text{Timestamp}
\end{aligned}
$$

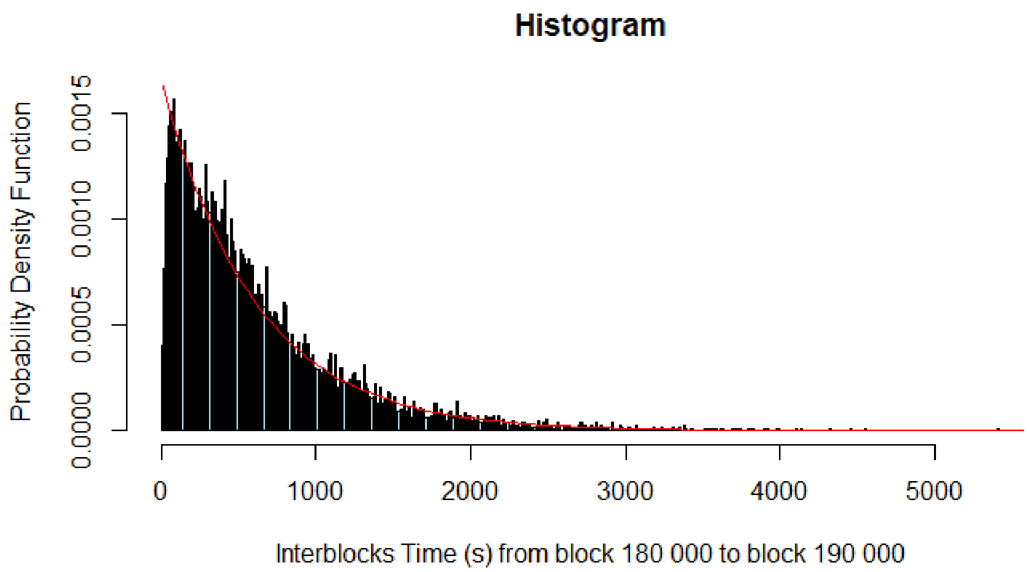
Block Header $= A|D|x$. Difficulty adjusted such that the time of resolution is $\approx$600 sec.

Example. Hash Genesis block & Block 500000

000000000019d6689c085ae165831e934ff763ae46a2a6c172b3f1b60a8ce26f
00000000000000000024fb37364cbf81fd49cc2d51c09c75c35433c3a1945d04

Blocksci (Princeton) github.com/citp/BlockSci
Open-source software platform for Blockchain analysis

Example. Between block 180000 and block 190000



However in general, KS & CVM tests fail...

Mathematics of mining

The time it takes to mine a block is memoryless

$$\mathbb{P}[T > t_1 + t_2 | T > t_2] \;=\; \mathbb{P}[T > t_1]$$

Proposition. *The random variable $\boldsymbol{T}$ has the exponential distribution with parameter $\alpha = \frac{1}{600}$ i.e.,*

$$f_{\boldsymbol{T}}(t) \;=\; \alpha\, \mathrm{e}^{-\alpha t}$$

Parameter α seen as a mining speed, $\mathbb{E}[\boldsymbol{T}] = \frac{1}{\alpha}$.

Definition. *Let $\boldsymbol{N}(t)$ be the number of blocks already mined at t-time. Start is at $t = 0$.*

Proposition. *The random process $\boldsymbol{N}$ is a Poisson process with parameter α i.e.,*

$$\mathbb{P}[\mathbf{N}(t) = k] \;=\; \frac{(\alpha\, t)^k}{k!}\,\mathrm{e}^{-\alpha t}$$

Notation. *Two group of miners. The letters $\boldsymbol{T}, \alpha, \mathbf{S}_n$, $\boldsymbol{N}$ (resp. $\boldsymbol{T}', \alpha', \mathbf{S}'_n, \mathbf{N}$) are reserved for honest miners (resp. attacker).*

Proposition. *Let $p := \mathbb{P}[\mathbf{T} < \boldsymbol{T}']$ and $q = 1 - p$. Then,*

$$\alpha \;=\; \frac{p}{\tau_0}$$

$$\alpha' \;=\; \frac{q}{\tau_0}$$

with $\tau_0 = 600$ sec.

Classical Double Spend Attack

No eclips attack (kind of Sybill's attack)

What is a double spend?

A single output may not be used as an input to multiple transactions.

- $T = 0$. A merchant **M** receives a transaction **tx** from **A** (= attacker). Transaction **tx** is issued from an UTXO **tx0**
- Honest Miners start mining openly, transparently
- Attacker **A** starts mining secretly
- One block of honest miners include **tx**
- No block of attacker include **tx**
- On the contrary, one blocks of the attacker includes another transaction **tx'** conflicting with **tx** from same UTXO **tx0**
- As soon as the z-th block has been mined, **M** sends his good to **A**
- **A** keeps on mining secretly
- As soon as A has mined a blockchain with a lenght greater than the official one, A broadcast his blockchain to the network
- Transaction **tx** has disappeared from the official blockchain.

Free Lunch!

Nakamoto's Analysis

Some definitions

Definition. *Let* $n \in \mathbb{Z}$*. We denote by* q_n *the probability of the attacker* $\boldsymbol{A}$ *to catch up honest miners whereas* $\boldsymbol{A}$*'s blockchain is* n *blocks behind.*

Then, $q_n = \left(\frac{q}{p}\right)^n$ if $n \geqslant 0$ and $q_n = 1$ else.

Definition. *For,* $z \in \mathbb{N}$*, the probability of success of a double-spending attack is denoted by* $P(z)$*.*

Note. The probability $P(z)$ is evaluated at $t = 0$. The double-spending attack cannot be successful before $t = \mathbf{S}_z$.

Formula for $P(z)$

When $t = \mathbf{S}_z$, the attacker has mined $\boldsymbol{N}'(\boldsymbol{S}_z)$ blocks. By conditionning on $\boldsymbol{N}'(\boldsymbol{S}_z)$, we get:

$$\begin{aligned}
P(z) &= \sum_{k=0}^{\infty} \mathbb{P}[\boldsymbol{N}'(\boldsymbol{S}_z) = k]\, q_{z-k} \\
&= \mathbb{P}[\boldsymbol{N}'(\boldsymbol{S}_z) \geqslant z] + \sum_{k=0}^{z-1} \mathbb{P}[\boldsymbol{N}'(\boldsymbol{S}_z) = k]\, q_{z-k} \\
&= 1 - \sum_{k=0}^{z-1} \mathbb{P}[\boldsymbol{N}'(\boldsymbol{S}_z) = k] \\
&\quad + \sum_{k=0}^{z-1} \mathbb{P}[\boldsymbol{N}'(\boldsymbol{S}_z) = k]\, q_{z-k} \\
&= 1 - \sum_{k=0}^{z-1} \mathbb{P}[\boldsymbol{N}'(\boldsymbol{S}_z) = k]\, (1 - q_{z-k})
\end{aligned}$$

Satoshi's approximation

White paper, Section 11 **Calculations**
According to Satoshi,

$$\boldsymbol{S}_z \approx \mathbb{E}[\boldsymbol{S}_z]$$

and

$$\begin{aligned}\boldsymbol{N}'(\boldsymbol{S}_z) &\approx \boldsymbol{N}'(\mathbb{E}[\boldsymbol{S}_z]) \\ &\approx \boldsymbol{N}'(z \cdot \mathbb{E}[\boldsymbol{T}]) \\ &\approx \boldsymbol{N}'\left(z \cdot \frac{\tau_0}{p}\right)\end{aligned}$$

So, $\boldsymbol{N}'(\boldsymbol{S}_z) \approx$ Poisson process with parameter λ given by

$$\begin{aligned}\lambda &= \alpha' \cdot z \cdot \frac{\tau_0}{p} \\ &= z \cdot \frac{q}{p}\end{aligned}$$

Definition. *We denote by $P_{\mathrm{SN}}(z)$ the (false) formula obtained by Satoshi in Bitcoin's white paper.*

Then,

$$P_{\mathrm{SN}}(z) = 1 - \sum_{k=0}^{z-1} \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}\left(1 - \left(\frac{q}{p}\right)^{z-k}\right)$$

However, $P(z) \neq P_{\mathrm{SN}}(z)$ since $\boldsymbol{N}'(\boldsymbol{S}_z) \neq \boldsymbol{N}'(\mathbb{E}[\boldsymbol{S}_z])$.

A correct analysis of double-spending attack

Meni Rosenfeld's correction

Set $\boldsymbol{X}_n := \mathbf{N}'(\boldsymbol{S}_n)$.

Proposition. *The random variable* $\boldsymbol{X}_n$ *has a negative binomial distribution with parameters* (n, p)*, i.e., for* $k \geqslant 0$

$$\mathbb{P}[\boldsymbol{X}_n = k] \; = \; p^n\, q^k \binom{k+n-1}{k}$$

"The attacker's potential progress" is not "a Poisson distribution with expected value $\lambda = z\,\frac{q}{p}$"...

Proposition. *The probability of success of a double-spending attack is*

$$P(z) \; = \; 1 - \sum_{k=0}^{z-1} (p^z\, q^k - q^z\, p^k) \binom{k+z-1}{k}$$

Numerical Applications

For $q = 0.1$,

z	$P(z)$	$P_{\text{SN}}(z)$
0	1	1
1	0.2	0.2045873
2	0.0560000	0.0509779
3	0.0171200	0.0131722
4	0.0054560	0.0034552
5	0.0017818	0.0009137
6	0.0005914	0.0002428
7	0.0001986	0.0000647
8	0.0000673	0.0000173
9	0.0000229	0.0000046
10	0.0000079	0.0000012

For $q = 0.3$,

z	$P(z)$	$P_{\text{SN}}(z)$
0	1	1
5	0.1976173	0.1773523
10	0.0651067	0.0416605
15	0.0233077	0.0101008
20	0.0086739	0.0024804
25	0.0033027	0.0006132
30	0.0012769	0.0001522
35	0.0004991	0.0000379
40	0.0001967	0.0000095

Solving for P less than 0.1%:

q	z	z_{SN}
0.1	6	5
0.15	9	8
0.20	18	11
0.25	20	15
0.3	32	24
0.35	58	41
0.40	133	89

Satoshi underestimates $P(z)$...

A closed form formula

Definition. *The **incomplete Beta function** is defined for* $a, b > 0$ *and* $x \in [0, 1]$ *by*

$$B_x(a, b) \ := \ \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{dt}$$

*The **regularized Beta function** is defined by*

$$I_x(a, b) \ := \ \frac{B_x(a, b)}{B_1(a, b)}$$

Classical result: for $a, b > 0$, $B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$

Theorem. *We have:*

$$P(z) \ = \ I_s(z, 1/2)$$

with $s = 4\,p\,q < 1$*.*

Proof. It turns out that the cumulative distribution function of a negative binomial random variable $\boldsymbol{X}$ (same notation as above) is

$$\begin{aligned} F_{\boldsymbol{X}}(k) &= \mathbb{P}[\boldsymbol{X} \leqslant k] \\ &= 1 - I_p(k+1, z) \end{aligned}$$

By parts,

$$I_p(k, z) - I_p(k+1, z) \ = \ \frac{p^k\, q^z}{k\, B(k, z)}$$

So,

$$P(z) \ = \ 1 - I_p(z, z) + I_q(z, z)$$

Classical symmetry relation for Beta function:

$$I_p(a,b)+I_q(b,a) \;=\; 1$$

(change of variable $t \mapsto 1-t$ in the definition). So,

$$I_p(z,z)+I_q(z,z) \;=\; 1$$

We also use:

$$I_q(z,z) \;=\; \frac{1}{2}\,I_s(z,1/2)$$

with $s=4\,p\,q$. □

Classical function pbeta implemented in R gives the true double-spending attack success probability.

Asymptotic analysis

According to Satoshi,

> Given our assumption that $p > q$, the probability drops exponentially as the number of blocks the attacker has to catch up with increases.

Theorem. *When $z \to \infty$, we have:*

$$P(z) \;\sim\; \frac{s^z}{\sqrt{\pi(1-s)\,z}}$$

with $s=4\,p\,q<1$.

A more accurate risk analysis

The merchant waits for z blocks. Once it has been done, he knows how long it took... Denote this number by τ_1. In average, it should take $\mathbb{E}[z\boldsymbol{T}] = \frac{z\,\tau_0}{p}$.

Definition. *Set* $\kappa := \frac{p\,\tau_1}{z\,\tau_0}$

Dimensionless parameter.

Satoshi's approximation: $\kappa = 1$...

Instead of computing $P(z)$, let us compute $P(z, \kappa)$.

Probability for a successful double-spending attack knowing that z blocks have been mined by the honest miners at $\boldsymbol{S}_z = \tau_1$.

Note. We have $P_{\mathrm{SN}}(z) = P(z, 1)$.

Note. Two different probabilities.

- Theoretical probability $P(z)$ calculated at $T = 0$ by the attacker or the merchant.

- concrete probability $P(z, \kappa)$ calculated at $T = \tau_1$ by the merchant .

Number of bocks mined by the attacker at $T = \tau_1$ unknown to the merchant = Poisson distribution parameter $\lambda(z,\kappa)$:

$$\begin{aligned}
\lambda(z,\kappa) &= \alpha'\,\tau_1 \\
&= \frac{q}{\tau_0}\cdot\frac{z\,\kappa\,\tau_0}{p} \\
&= \frac{z\,q}{p}\,\kappa
\end{aligned}$$

i.e.,

$$\mathbb{P}[\boldsymbol{N}'(\tau_1) = k] \;=\; \frac{\left(\frac{z\,q}{p}\,\kappa\right)^k}{k!}\,\mathrm{e}^{-\frac{z\,q}{p}\kappa}$$

Definition. *The regularized Gamma function is defined by:*

$$\Gamma(s,x) \;:=\; \int_x^{+\infty} t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{dt}$$

The regularized incomplete Gamma function is:

$$Q(s,x) \;:=\; \frac{\Gamma(s,x)}{\Gamma(s)}$$

It turns out that

$$Q(z,\lambda) \;=\; \sum_{k=0}^{z-1} \frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}$$

So,

Theorem. *We have:*

$$P(z,\kappa) \;=\; 1 - Q\left(z, \frac{\kappa\,z\,q}{p}\right) + \left(\frac{q}{p}\right)^z \mathrm{e}^{\kappa z \frac{p-q}{p}}\,Q(z,\kappa\,z)$$

Proof. We have:

$$\begin{aligned} P(z,\kappa) &= \mathbb{P}[\boldsymbol{N}'(\tau_1)\geqslant z]+\sum_{k=0}^{z-1}\mathbb{P}[\boldsymbol{N}'(\tau_1)=k]\,q_{z-k} \\ &= 1-\sum_{k=0}^{z-1}\frac{\lambda(z,\kappa)^k}{k!}\,\mathrm{e}^{-\lambda(z,\kappa)} \\ &+ \sum_{k=0}^{z-1}\left(\frac{q}{p}\right)^{z-k}\cdot\frac{\lambda(z,\kappa)^k}{k!}\,\mathrm{e}^{-\lambda(z,\kappa)} \\ &= 1-Q\left(z,\frac{\kappa\,z\,q}{p}\right)+\left(\frac{q}{p}\right)^z\mathrm{e}^{\kappa z\frac{p-q}{p}}\,Q(z,\kappa\,z) \end{aligned}$$

□

Asymptotics Analysis

Proposition. *We have* $P_{\mathrm{SN}}(z)\sim\frac{\mathrm{e}^{-z\,c\left(\frac{q}{p}\right)}}{2}$ *with*

$$c(\mu)\ :=\ \mu-1-\ln\mu$$

More generally, we have 5 different regimes.

Proposition 1. *When* $z\to+\infty$, *we have:*

- *For* $0<\kappa<1$, $P(z,\kappa)\sim\frac{1}{1-\kappa\,\frac{q}{p}}\frac{1}{\sqrt{2\,\pi\,z}}\,\mathrm{e}^{-zc\left(\kappa\frac{q}{p}\right)}$
- *For* $\kappa=1$, $P(z,1)=P_{\mathrm{SN}}(z)\sim\frac{\mathrm{e}^{-z\,c\left(\frac{q}{p}\right)}}{2}$
- *For* $1<\kappa<\frac{p}{q}$,

$$P(z,\kappa)\sim\frac{\kappa\left(1-\frac{q}{p}\right)}{(\kappa-1)\left(1-\kappa\,\frac{q}{p}\right)}\frac{1}{\sqrt{2\,\pi\,z}}\,\mathrm{e}^{-zc\left(\kappa\frac{q}{p}\right)}$$

- *For* $\kappa = \frac{p}{q}$, $P\left(z, \frac{p}{q}\right) \to \frac{1}{2}$ *and*

$$P\left(z, \frac{p}{q}\right) - \frac{1}{2} \sim \frac{1}{\sqrt{2\,\pi\,z}} \left(\frac{1}{3} + \frac{q}{p-q}\right)$$

- *For* $\kappa > \frac{p}{q}$, $P(z, \kappa) \to 1$ *and*

$$1 - P(z, \kappa) \sim \frac{\kappa\left(1 - \frac{q}{p}\right)}{\left(\kappa\frac{q}{p} - 1\right)(\kappa - 1)} \frac{1}{\sqrt{2\,\pi\,z}}\, \mathrm{e}^{-z c\left(\kappa\frac{q}{p}\right)}$$

Comparison between $P(z)$ and $P_{\mathrm{SN}}(z)$

Asymptotic behaviours

The asymptotic behaviours of $P(z)$ and $P_{\mathrm{SN}}(z)$ are quite different

Proposition. *We have* $P_{\mathrm{SN}}(z) \prec P(z)$

Bounds for $P(z)$ and $P_{\mathrm{SN}}(z)$

Goal: compute an explicit rank z_0 such that

$$P_{\mathrm{SN}}(z) \;<\; P(z)$$

for all $z > z_0$.

Upper and lower bounds for $P(z)$

Remember that $s = 4\,p\,q$.
We'll use Gautschi's inequalities.

Proposition 2. *For any* $z > 1$,

$$\sqrt{\frac{z}{z+1}}\,\frac{s^z}{\sqrt{\pi\, z}} \leqslant P(z) \leqslant \frac{s^z}{\sqrt{\pi\,(1-s)\,z}}$$

An upper bound for $P_{\mathrm{SN}}(z)$

Lemma. *Let* $z \in \mathbb{N}^*$ *and* $\lambda \in \mathbb{R}_+^*$. *We have:*

i. *If* $\lambda \in]0,1[$, *then*

$$1 - Q(z, \lambda\, z) \;<\; \frac{1}{1-\lambda}\,\frac{1}{\sqrt{2\,\pi\, z}}\,\mathrm{e}^{-z(\lambda - 1 - \ln \lambda)}$$

ii. *If* $\lambda = 1, Q(z,z) < \frac{1}{2}$.

Proposition. *We have*

$$P_{\mathrm{SN}}(z) \;<\; \frac{1}{1-\frac{q}{p}}\,\frac{1}{\sqrt{2\,\pi\, z}}\,\mathrm{e}^{-z c\left(\frac{q}{p}\right)} + \frac{1}{2}\,\mathrm{e}^{-z c\left(\frac{q}{p}\right)}$$

with $c(\mu) := \mu - 1 - \ln \mu$.

An explicit rank z_0

Theorem. *Let* $z \in \mathbb{N}^*$. *A sufficient condition to get* $P_{\mathrm{SN}}(z) < P(z)$ *is* $z > z_0$ *with*

$$z_0 \;:=\; \mathrm{Max}\left(\frac{2}{\pi\left(1-\frac{q}{p}\right)^2}, \frac{1}{2\sqrt{2}} - \frac{1+\frac{1}{\sqrt{2}}}{2}\,\frac{\ln\left(\frac{2\,\psi_0}{\pi}\right)}{\psi_0}\right)$$

with

$$\psi_0 \;:=\; \frac{q}{p} - 1 - \ln\left(\frac{q}{p}\right) - \ln\left(\frac{1}{4\,p\,q}\right) > 0$$

Conclusion. What sould the merchant do?

Set $\bar{P}(z,t) =$ probability of success of a double spend attack knowing that z blocks have been validated before t-date.

Shipment condition: Good will be sent to the buyer as soon as $\bar{P}(z,t) < 0.1\%$ for any $q < 0.2$ (for instance) where $t =$ time used to mine z blocks and
cf Satoshi Risk Tables.

Shipment_time = $\mathrm{Inf}\{t > 0 \,/\, \bar{P}(N(t),t) < \varepsilon\}$.

On average, this will happen after z blocks have been validated and $P(z) < \varepsilon$.

Proposition. *One has* $P(z) = \mathbb{E}[P(z, \boldsymbol{\kappa})]$.

and $\boldsymbol{\kappa} := \frac{p\,S_z}{z\,\tau_0}$ as above.

So, by Markov inequality,

$$\forall \varepsilon > 0, \quad \mathbb{P}[P(z,\boldsymbol{\kappa}) > \varepsilon] \;<\; \frac{P(z)}{\varepsilon} \;\to\; 0$$

Note. If $\kappa > 1$, $\mathbb{P}[\boldsymbol{\kappa} > \kappa] \sim \frac{1}{\kappa - 1}\frac{1}{\sqrt{2\,\pi\,z}}\,\mathrm{e}^{-zc(\kappa)}$. Other asymptotics in DSR.

So, $\mathbb{P}[\text{Shipment\_Time} < +\infty] = 1$.

Submissions

Long list of rejections from

- arxiv.org (section probability)
- European Journal of Operational Research: "I came to conclusion that your paper does not fit the scope of EJOR. Your list of references also gives support to this conclusion.", Emanuele Borgonovo
- Acta Informatica: "[...]the list of references, [...] is comparably short and does not refer to any paper of the typical Acta Informatica areas.", Christel Baier
- SIAM Journal on Financial Mathematics: "Overall, the authors basically just recast really basic probability results using bitcoin jargons. I think rejection outright is the right decision.", Jean-Pierre Fouque
- Journal of Economic Theory: "The paper does not contribute to any ongoing conversations in economics.", Laura Veldkamp

Finally submitted to International Journal of Theoretical and Applied Finance: "We will send the paper to referees and the process will take approximately 5-6 months."